Completeness of MLL proof-nets w.r.t. weak distributivity

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Abstract. We examine 'weak-distributivity' as a rewriting rule $\overset{\mathsf{WD}}{\rightsquigarrow}$ defined on multiplicative proof-structures (so, in particular, on multiplicative proof-nets: MLL). This rewriting does not preserve the type of proofs-nets, but does nevertheless preserve their correctness. The specific contribution of this paper, is to give a direct proof of completeness for $\overset{\mathsf{WD}}{\rightsquigarrow}$: starting from a set of simple generators (proof-nets which are a n-ary $\otimes$ of $⅋$-ized axioms), any mono-conclusion MLL proof-net can be reached by $\overset{\mathsf{WD}}{\rightsquigarrow}$ rewriting (up to $\otimes$ and $⅋$ associativity and commutativity).

1 Preliminaries

1.1 Multiplicative Linear Logic: sequent calculus and proof-nets

The formulas of Multiplicative Linear Logic [1] are defined from the following grammar:

$$\underbrace{\mathsf{A} = X, X^{\perp}, Y, Y^{\perp}, \dots}_{\text{Atoms}} \;\Big|\; \underbrace{\mathsf{A} \otimes \mathsf{A}}_{\text{Tensor}} \;\Big|\; \underbrace{\mathsf{A} ⅋ \mathsf{A}}_{\text{Par}}$$

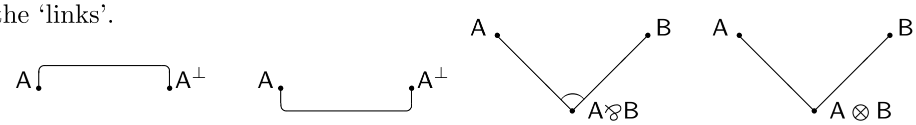
Negation ("orthogonal") $(.)^{\perp}$ is not a connective, but a defined unary operation over formulas, inductively defined by:

$$(X)^{\perp} = X^{\perp},\ (X^{\perp})^{\perp} = X,\ (\mathsf{A} \otimes \mathsf{B})^{\perp} = \mathsf{A}^{\perp} ⅋ \mathsf{B}^{\perp},\ (\mathsf{A} ⅋ \mathsf{B})^{\perp} = \mathsf{A}^{\perp} \otimes \mathsf{B}^{\perp}$$

A *sequent* is a multiset Γ of formulas, written $\vdash \Gamma$. The rules of sequent calculus MLL (from which MLL sequent calculus derivations is inductively defined as usual) are :

$$\underset{\text{(Identity axiom)}}{\frac{}{\vdash \mathsf{A}, \mathsf{A}^{\perp}}} \qquad \mathsf{cut}\frac{\vdash \Gamma, \mathsf{A} \qquad \vdash \Delta, \mathsf{A}^{\perp}}{\vdash \Gamma, \Delta} \qquad ⅋\frac{\vdash \Gamma, \mathsf{A}\ ,\ \mathsf{B}}{\vdash \Gamma, \mathsf{A} ⅋ \mathsf{B}} \qquad \otimes\frac{\vdash \Gamma, \mathsf{A} \qquad \vdash \Delta, \mathsf{B}}{\vdash \Gamma, \Delta, \mathsf{A} \otimes \mathsf{B}}$$

Definition 1 Let A, B be any formulas. The following four minigraphs are called the 'links'.



axiom-link cut-link ⅋-link ⊗-link

In those minigraphs, edges are intended to be oriented: in both directions for axiom-links and cut-links, and following the natural downward orientation of space for the others (the curve edge in the ⅋-link is just an indication used later). A formula occurrence (vertex) which is a target (resp. a source) of an (oriented) edge in a link is a 'conclusion' (resp. a 'premise') of that link if it is reached downward (resp. if it is the source of a downward beginning edge).

Definition 2 A *structure* is a graph whose vertices are (labeled with) formulas inductively built by using the rules below:

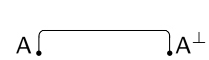
1. links are structures,
2. structures are closed by multiset-union,
3. structures are closed by identification (in a structure) of occurrences of (vertices labeled with) a same formula, if this identification preserves the fact that any formula occurrence is conclusion of at most one link and premise of at most one link.

The top (resp. bottom) formulas of a structure are called its hypothesis (resp. conclusions). If

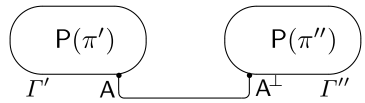
In the sequel, we will mainly work with the cut-free fragment of the set of proof-structures. Also, we will only represent, in structures, the formulas being conclusion of axiom-links (see for instance the proof-strucure called $\overline{\mathsf{WD}}$, represented in section 2). Indeed, taking account of the mark put on ⅋-links (the curve edge), this information is sufficient for recovering all missing formulas (in the cut free fragment, the ambiguity occasioned, when some formulas are so missing, by commutativity, is harmless).

Let π a sequent calculus proof in MLL. Let $\mathsf{P}(\pi)$ be the structure (with same conclusions as π) obviously defined by recurrence on π construction as presented below:

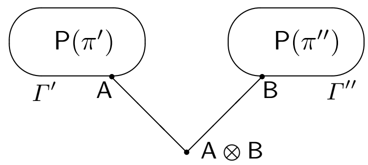
$$\overline{\vdash \mathsf{A}\,,\ \mathsf{A}^{\perp}} \qquad \longmapsto$$



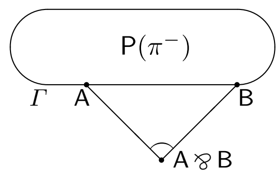
$$\dfrac{\dfrac{\vdots\ \pi'}{\vdash \Gamma', \mathsf{A}^{\perp}} \qquad \dfrac{\vdots\ \pi''}{\vdash \Gamma'', \mathsf{A}}}{\vdash \Gamma', \Gamma''} \quad \longmapsto$$



$$\dfrac{\dfrac{\vdots\ \pi'}{\vdash \Gamma', \mathsf{A}} \qquad \dfrac{\vdots\ \pi''}{\vdash \Gamma'', \mathsf{B}}}{\vdash \Gamma', \Gamma'', \mathsf{A} \otimes \mathsf{B}} \quad \longmapsto$$



$$\dfrac{\dfrac{\vdots\ \pi^{-}}{\vdash \Gamma, \mathsf{A}\ ,\ \mathsf{B}}}{\vdash \Gamma, \mathsf{A} \parr \mathsf{B}} \quad \longmapsto$$



Definition 3 The set of 'multiplicative proof-nets' (still noted MLL) is the range of $\mathsf{P}(.)$.

Let us recall that $\mathsf{P}(.) : \mathsf{MLL} \longrightarrow \{\text{structures}\}$ is neither injective (reason why it is interesting), nor surjective (reason why it is a little bit difficult to deal with proof-nets). And indeed, in the prehistory of proof-nets, to find an intrinsic characterization of proof-nets among structures (standard old "sequentialization" problem) was the first question to solve :

Definition 4 A structure S is sequentializable, if $\mathsf{P}(\pi) = \mathsf{S}$ for some sequent derivation $\pi \in \mathsf{MLL}$.

In this paper, we will deeply use sequentialization tools worked out by Girard [1] and Danos and Regnier [2], and in particular:

Definition 5 Let S be a structure.

- A *switching* in S is any sub-graph of S one gets by erasing one edge in any $\parr$-link in S.
- A structure S satisfies the Danos-Regnier criterion ($\mathsf{S} \Vdash DR$) if any of its switchings is acyclic and connected.
- A structure S (possibly) with hypothesis, such that $\mathsf{S} \Vdash DR$, is called a *module*.

Proposition 6 (Danos-Regnier) Let S an hypothesis free structure.

$$\mathsf{S} \text{ is sequentializable} \quad \text{iff} \quad \mathsf{S} \Vdash DR$$

1.2 Pretypes of modules and orthogonality

In this subsection a few tools and results used later are recalled. They all come from Danos thesis (see [3]).

Definition 7 The "border" of a structure S is the multiset of its hypothesis and conclusions (notation: $\mathsf{Border}(\mathsf{S})$).

For sake of simplicity, every time no attention is needed to the specific formulas being conclusion or hypothesis of the structures under consideration, and in particular in the present subsection, we will forget them, then just using indices (integers) to describe their border.

Definition 8 Let S a structure and σ a switching of S. The partition of $\mathsf{Border}(\mathsf{S})$ induced by σ, is the quotient of $\mathsf{Border}(\mathsf{S})$ by the relation defined by: "*n is (in σ) in the same connected component as m*".

To note a given partition of, say, $\{1,2,3,4,5\}$, for instance $\big\{\{1\},\{2,4\},\{3,5\}\big\}$, we will use the simplified notation: $\underline{1}\ \underline{24}\ \underline{35}$.

Definition 9 The pretype[1] of S is the set P_{S} of all partitions induced over $\mathsf{Border}(\mathsf{S})$ by all switchings of S.

Definition 10 If $\mathsf{Border}(\mathsf{S}) = \{1,\dots,n\}$ and $\mathsf{Border}(\mathsf{S}') = \{1',\dots,n'\}$, we note $\mathsf{S} :: \mathsf{S}'$ the graph resulting of the plugging of S and S' together via the border (identifying vertex i with i').

Definition 11 The *meeting graph* $\mathcal{G}(p,q)$ of partitions p,q over $\{1,\dots,n\}$, is the graph whose set of vertices is $p \uplus q$, and such that one puts one edge between a class in p and a class in q for each point they share.

Examples Meeting graphs of partitions over $\{1,2,3,4,5\}$

$$\begin{array}{ccc} p & = & \underline{1}\ \ \underline{23}\ \ \underline{45} \\ & & |\ /\ |\ /\ | \\ q & = & \underline{12}\ \ \underline{34}\ \ \underline{5} \end{array} \qquad\qquad \begin{array}{ccc} r & = & \underline{123}\ \ \ \ \underline{45} \\ & & ()\ \backslash\ /\ | \\ q & = & \underline{12}\ \ \underline{34}\ \ \underline{5} \end{array}$$

$$\mathcal{G}(p,q) \qquad\qquad\qquad \mathcal{G}(r,q)$$

Definition 12 p is orthogonal to q (notation: $p \perp q$), if $\mathcal{G}(p,q)$ is acyclic and connected (so in the example above: $p \perp q$, but $r \not\perp q$).

[1] I follow Maieli and Puite who, in [4], call 'pretype' of a module what Danos calls in [3] its 'type' (a terminology that the former reserve to the double orthogonal of the 'type' in the sense of Danos)

Definition 13 Two sets P and Q of partitions over $\{1,\ldots,n\}$ are orthogonal, if they are pointwise orthogonal (notation: $P \perp Q$)

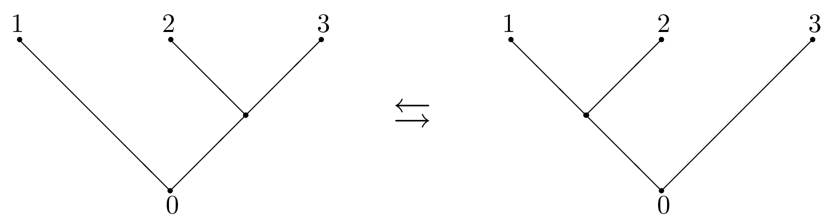
Definition 14 $P^{\perp} = \{q\ ;\ \forall p \in P, q \perp p\}$

Remark 1 $P \perp Q \quad \Rightarrow \quad P \subseteq Q^{\perp}$

Proposition 15 (Danos) $\mathsf{S} :: \mathsf{S}'$ is a proof-net $\Leftrightarrow$ $P_{\mathsf{S}} \perp P_{\mathsf{S}'}$

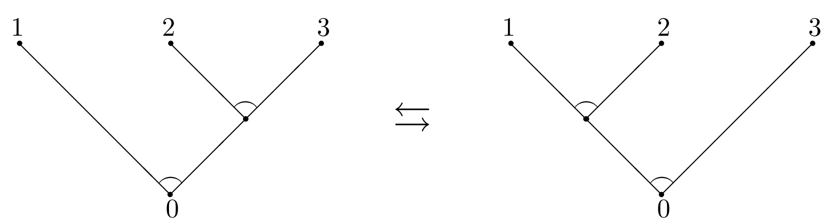
Examples

1. $\otimes$-associativity:



The two modules above have same pretype: $\{\,\underline{0123}\,\}$. We thus can replace in a given proof-net any sub-graph like one of those above by the other one: although such a replacement changes the type of the proof-net, by proposition 15, the satisfaction of the DR-correctness criterion is preserved.

2. $\parr$-associativity:



The two modules above have same pretype: $\{\,\underline{01}\ \underline{2}\ \underline{3}\,,\ \underline{1}\ \underline{02}\ \underline{3}\,,\ \underline{1}\ \underline{2}\ \underline{03}\,\}$. Same remark as above.

(To put in in one slogan: "Pretype equality[2] = free associativity".)

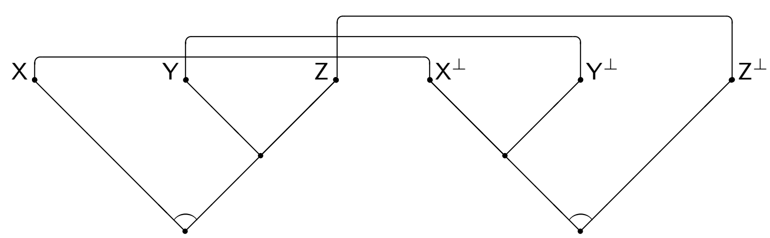
Remark As far as *commutative* MLL is in concern, structures differing only up to commutativity are not distinguished. This means however, that when one plugs S and S' together to build $\mathsf{S}::\mathsf{S}'$, one has to pay attention to continue to identify same indices of the border.

[2] Notice that the equality of the pretypes, in both case above, is not affected by the commutativity of the connective respectively involved

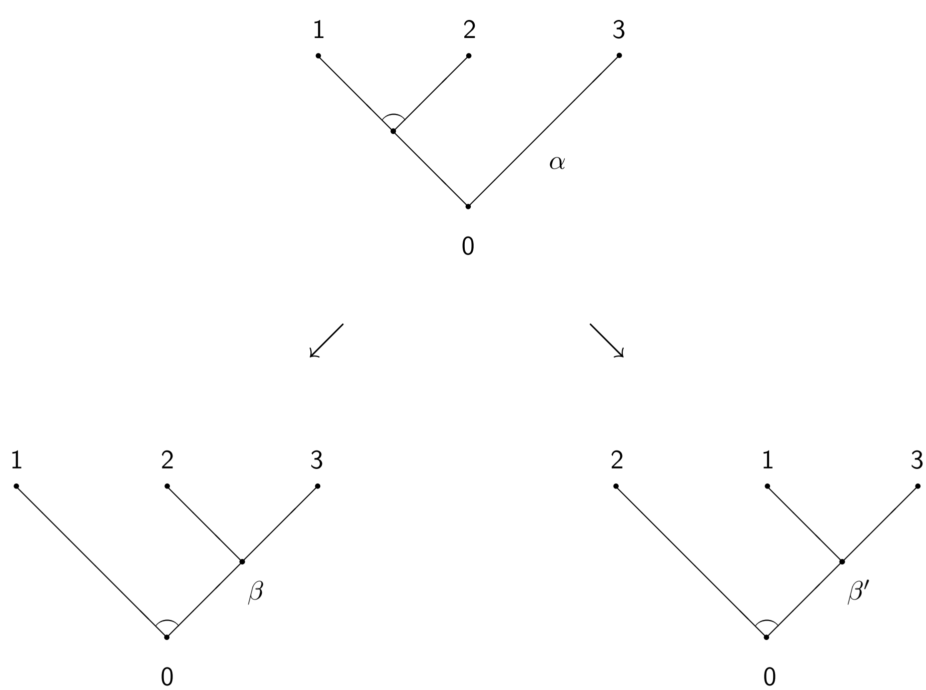
2 Weak-distributivity: a proof-net, a computational morphism

The formula $(\mathsf{A} \vee \mathsf{B}) \wedge \mathsf{C} \;\rightarrow\; \mathsf{A} \vee (\mathsf{B} \wedge \mathsf{C})$ usually called the *weak-distributivity* formula (WD), is a theorem of LK (sequent calculus for classical logic) and even actually of its fragment just presented: multiplicative linear logic. Indeed, the multiplicative version of WD, the sequent $(\mathsf{A} \parr \mathsf{B}) \otimes \mathsf{C} \vdash \mathsf{A} \parr (\mathsf{B} \otimes \mathsf{C})$ or its one-sided avatar $\vdash (\mathsf{A}^\perp \otimes \mathsf{B}^\perp) \parr \mathsf{C}^\perp, \mathsf{A} \parr (\mathsf{B} \otimes \mathsf{C})$ are derivable in the corresponding sequent calculi.

It is easy to see that in MLL proof-nets syntax, there is a unique cut free net $\overline{\mathsf{WD}}$ with conclusions $\mathsf{X} \parr (\mathsf{Y} \otimes \mathsf{Z}), (\mathsf{X}^\perp \otimes \mathsf{Y}^\perp) \parr \mathsf{Z}^\perp$ $(\mathsf{X}, \mathsf{Y}, \mathsf{Z}$ distinct atoms). Here it is:



Remark 2 One can use (instanciations of) $\overline{\mathsf{WD}}$ as 'surgical morphisms' to replace (through the cut-elimination process), in any cut-free proof-net, any subgraph of shape α by either the module β or the module β' (all pictured below), anything else remaining unchanged.



Proof Just observe the effect of cut elimination, when $\mathsf{N_{WD}}$ (or, to be precise, $\mathsf{N_{WD}}[\mathsf{A}/\mathsf{X}, \mathsf{B}/\mathsf{Y}, \mathsf{C}/\mathsf{Z}]$) is cut against a proof-net with a terminal α, thus corresponding to a conclusion $(\mathsf{A} \parr \mathsf{B}) \otimes \mathsf{C}$. Naturally, the module α being in general 'deep' (i.e. not terminal) in the proof-net, one possibly needs, previously, to complete $\mathsf{N_{WD}}$ downwards, in the obvious way, by identities. A detailed proof would require a long presentation of notions faraway from the main subject of the paper - it should be required in particular to define the elementary cut-elimination steps in MLL and, in order to make precise this "completion of $\overline{\mathsf{WD}}$ by identities", to define the replacement of a given identity-axiom by $\mathsf{N_{WD}}$ in identity-axioms partial η-expansions (see [5]). As remark 2 is not crucial for the rest of the paper (it is only used as an *alternative* proof for proposition 17 below), we will leave it at that. ⊠

Nota: in MLL, cut-elimination is deterministic (so for a given instance of a cut-link, a unique reduction step applies); so the apparent non determinism comes here from the fact that, proof-nets being defined up to commutativity, the spatial representation of binary links ($\otimes$ and $\parr$-link) does not uniquely determine their conclusion.

We now internalize as a rewriting rule the transformation realized by $\overline{\mathsf{WD}}$ under cut-elimination.

3 Weak-distributivity as a sound and complete rewriting

3.1 The completeness problem

Definition 16 Let $\overset{\mathsf{WD}}{\rightsquigarrow}_1$ be the smallest binary relation over the set of structures including $\{(\alpha, \beta), (\alpha, \beta')\}$ and compatible with structures construction. Let $\overset{\mathsf{WD}}{\rightsquigarrow}$ denotes the transitive and reflexive closure of $\overset{\mathsf{WD}}{\rightsquigarrow}_1$, a.k.a. the rewriting rule over structures generated by $\overset{\mathsf{WD}}{\rightsquigarrow}_1$.

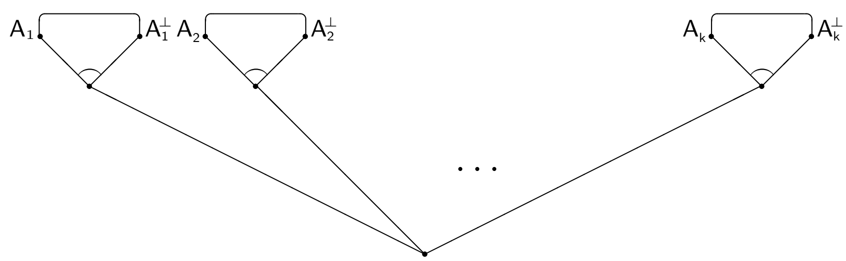
Notice that $\overset{\mathsf{WD}}{\rightsquigarrow}_1$ rewriting does not preserve the type (the proof-net conclusions), nevertheless:

Proposition 17 The set of MLL proof-nets is closed under $\overset{\mathsf{WD}}{\rightsquigarrow}$ rewriting.

Proof By remark 2 and closure of MLL proof-nets under cut-elimination. This could as well be proved by means of the 'Pretypes' technics presented in subsection 1.2 (as Maieli and Puite did in [4]). ⊠

A natural question then is whether any proof-net can be obtained that way from a relevant basic set of proof-nets (*completeness problem*). Taking account of the kind of $\otimes/\parr$ permutations involved in $\overset{\text{WD}}{\rightsquigarrow}$ rewriting, a rather natural candidate for this set of generators is the set of proof-nets in which the $\parr$-links are 'as high as possible', the $\otimes$-links 'as low as possible'. To give a more compact representation of those proof-nets, it is useful to consider proof-nets up to $\otimes$ associativity, an harmless quotientation as far as conservation of proof-nets type is no more in concern (see subsection 1.2), and to use an k-ary $\otimes$-link for the representation:

Definition 18 A proof-net is in canonical form if it is (a representative) of the following form :



We will refer to *axiom links whose conclusions are both premises of a same $\parr$-link* (as above) as $\parr$-ized axioms. So a proof-net in canonical form is "a k-ary $\otimes$ of $\parr$-ized axioms".

So, to sum up within the 'canonical form' terminology, one knows from proposition 17, that:

Proposition 19 *(Soundness)* Any proof-structure generated from a proof-net in canonical form by $\overset{\text{WD}}{\rightsquigarrow}$ rewriting is a proof-net.

And the problem which remains to solve is the following:

Problem *(Completeness)* Can any proof-net be generated from some canonical proof-net by $\overset{\text{WD}}{\rightsquigarrow}$ rewriting ?

That question (or at least a similar one) has already been solved by categorical means in [6]. The aim of the present section, however, is to give an original, direct, combinatorial proof.

Notice that the naive idea which likely comes first in mind to prove completeness (namely by just reversing the rewriting) is wrong: the converse $\overset{\text{WD}}{\rightsquigarrow}{}_1^{-}$ of $\overset{\text{WD}}{\rightsquigarrow}{}_1$ rewriting does *not* generally preserve the correctness criterion : proof-nets are *not* closed under $\overset{\mathsf{WD}}{\rightsquigarrow}^{-}$ (the transitive and reflexive closure of $\overset{\mathsf{WD}}{\rightsquigarrow}^{-}_{1}$). Worst, there exists proof-nets for which any $\overset{\mathsf{WD}}{\rightsquigarrow}^{-}$ rewriting leads out of the set of proof-nets (examples listed at the beginning of subsection 3.3).

However, we are going to show that inasmuch one considers only proof-nets with a unique conclusion ('mono-conclusion' proof nets) and one works up to associativity and commutativity, then:

1. in a proof-net (not being in canonical form), *there exists* a sub-graph of shape β for which one of the corresponding $\overset{\mathsf{WD}}{\rightsquigarrow}^{-}$ replacements is correct;
2. some ordinal attached to proof-structures decreases when one performs such a replacement.

3.2 Doubly splitting $\substack{\otimes \\ \parr}$

Definition 20

1. Following Girard's terminology in [1], an instance of a $\otimes$-link in a (say, connected) structure S is a *splitting-*$\otimes$, if the erasure in S of that link (conclusion vertex and edges) produces two unconnected structures.
2. Following Danos terminology in [3], an instance of a $\parr$-link in a (say, connected) structure S is a *splitting-*$\parr$ (*un* $\parr$ *scindant*), if the erasure of the two edges of that link in S produces two unconnected graphs (we will then note the upper one - a structure - by S_+).
3. In a proof-structure S, a splitting $\parr$-link surmounted by a $\otimes$-link which itself splits the corresponding S_+, will be called a *doubly splitting* $\substack{\otimes \\ \parr}$.

Concerning proof-nets splittings, we will use soon the two following lemmas both picked up from the saga of proof-nets 'sequentialization' tools.

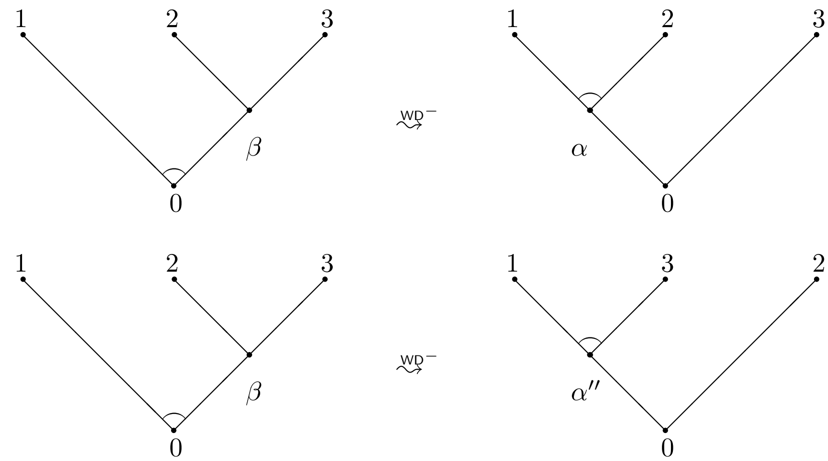
Lemma 21 (Girard, [1]) Let N a proof-net with no terminal $\parr$-link. If the set of $\otimes$-link in N is not empty, then (the set of terminal $\otimes$-link in N is not empty and) one of them is a splitting-$\otimes$.

Lemma 22 (Danos, [3] Let N a proof-net. If the set of $\parr$-link in N is not empty, one of them is a splitting-$\parr$.

Proposition 23 The $\overset{\mathsf{WD}}{\rightsquigarrow}^{-}$ rewriting rule preserves correctness when applied to a doubly splitting $\substack{\otimes \\ \parr}$.

Proof Let us consider again the module β met in remark 2 (the treatment of β' is similar). Up to $\parr$ and $\otimes$ commutativity, β has two images (α and α'') by

$\overset{\text{WD}}{\rightsquigarrow}^{-}$ as pictured below:



Let us calculate, for each module above, its pretype (the various partitions induced by switchings over its border $\{0,1,2,3\}$) and the orthogonal of this pretype. We have:

$$P_\beta = \{\underline{023}\ \underline{1}\ ,\ \underline{10}\ \underline{23}\}$$
$$P_\beta^{\perp} = \{\underbrace{\underline{0}\ \underline{12}\ \underline{3}}_{p_1}\ ,\ \underbrace{\underline{0}\ \underline{13}\ \underline{2}}_{p_2}\}$$

$$P_\alpha = \{\underline{023}\ \underline{1}\ ,\ \underline{013}\ \underline{2}\}$$
$$P_\alpha^{\perp} = \{p_1\}$$

$$P_{\alpha''} = \{\underline{023}\ \underline{1}\ ,\ \underline{012}\ \underline{3}\}$$
$$P_{\alpha''}^{\perp} = \{p_2\}$$

Let $\mathsf{N}[\beta]$ be a proof-net containing β as a sub-module. We note $\mathsf{N}[\ ]$ the complementary module to β in $\mathsf{N}[\beta]$ (a.k.a. the context). Because $\mathsf{N}[\,] :: \beta$ is a proof-net, we know that $P_{\mathsf{N}[\,]} \perp P_\beta$ (by proposition 15), and thus that $P_{\mathsf{N}[\,]} \subseteq P_\beta^{\perp}$ (by remark 1). From our computation just above, we thus have $P_{\mathsf{N}[\,]} \subseteq \{p_1, p_2\}$. However, by invoking now our computations for α (resp. α'') and the results just cited again, one sees that $\mathsf{N}[\alpha]$ (resp. $\mathsf{N}[\alpha'']$) is a proof-net only if $P_{\mathsf{N}[\,]} \subseteq \{p_1\}$ (resp. $P_{\mathsf{N}[\,]} \subseteq \{p_2\}$). In other words, applying $\overset{\text{WD}}{\rightsquigarrow}^{-}$ rewriting to $\mathsf{N}[\beta]$ is correct if and only if either $P_{\mathsf{N}[\,]} = \{p_1\}$ or $P_{\mathsf{N}[\,]} = \{p_2\}$.

But this is precisely what happens when the ⅋-link and the ⊗-link of β form together a doubly splitting $\overset{\otimes}{⅋}$ in $\mathsf{N}[\beta]$. ⊠

3.3 Existence of a correct $\overset{\text{WD}}{\rightsquigarrow}^{-}$ strategy

We will now show that it is always possible to find such a doubly splitting $\overset{\otimes}{⅋}$ situation. Such a statement happens to be false while one keeps the usual syntax unquotiented, but true when one considers only mono-conclusion nets (in terms of expressive power, nothing is lost: $\vdash_{\mathsf{MLL}} \Gamma$ iff $\vdash_{\mathsf{MLL}} \parr \Gamma$) and up to associativity and commutativity of $\otimes$ and $\parr$.

All of the three conditions: (1) *mono-conclusion* proof-nets, (2) *associativity*, and (at least some form of) (3) *commutativity* appear compulsory. Indeed, If we drop condition (n) (where $n \in \{1,2,3\}$), keeping the two others, the unique cut-free proof-net having for conclusions the formula(s) indicated in the corresponding item $[n]$ below is a counter-example (it does not include any sub-graph of shape β for which the corresponding $\overset{\mathsf{WD}^-}{\rightsquigarrow}$ replacement is correct):

$$(\mathsf{X} \parr \mathsf{Y}) \otimes \mathsf{Z}\,,\, \mathsf{Z}^{\perp} \parr (\mathsf{Y}^{\perp} \otimes \mathsf{X}^{\perp}) \quad \text{(mono-conclusion) [1]}$$
$$((\mathsf{X} \parr \mathsf{Y}) \otimes \mathsf{Z}) \parr (\mathsf{Z}^{\perp} \parr (\mathsf{Y}^{\perp} \otimes \mathsf{X}^{\perp})) \quad \text{(associativity) [2]}$$
$$\mathsf{X} \parr ((\mathsf{Y} \parr \mathsf{Y}^{\perp}) \otimes \mathsf{X}^{\perp}) \quad \text{(commutativity) [3]}$$

We now state the series of lemmas, combined hereafter to prove the existence of a correct $\overset{\mathsf{WD}^-}{\rightsquigarrow}$ strategy.

Lemma 24 Let N be a mono-conclusion cut-free proof-net. If the terminal link of N is a $\parr$-link whose premises (up to associativity) all are conclusions of identity-axioms, then N does not contain any $\otimes$-link.

Proof Else, let us consider a $\otimes$-link in N and p a maximal downward path in N beginning with one of the edges of that $\otimes$-link. N being mono-conclusion, the last edge of p must be one of the edges of the terminal $\parr$-link of N. Hence in p (sequences of) $\otimes$-link edges alternate at least once with (sequences of) $\parr$-link edges. Up to associativity, the terminal $\parr$-link of N thus should be surmounted by a $\otimes$-link. ⊠

To save time (by avoiding drawings), the proofs we give now (of lemmas 25 and 27) release upon sequentialization. This is of course just a matter of convenience.

Notation: if π is a sequent calculus derivation, we note $|\pi|$ the number of non 0-ary rules in π.

Lemma 25 Let N be a cut-free proof-net. If N does not contain any $\otimes$-link, then N is either an identity-axiom or a $\parr$-ized identity-axiom.

Proof By sequentialization theorem, it suffices to prove the lemma for cut-free sequent calculus. Let π an MLL derivation of $\vdash \Gamma$ and let r be the last rule of π. If $|\pi| = 0$, then r is an identity-axiom and we are done. Else, as there is no $\otimes$-rule, r has to be a $\parr$ rule. Let π^- the immediate sub-proof of π. By induction hypothesis (and the case as of a $\parr$-ized identity-axiom being impossible, π^-'s terminal sequent having more than one formula, namely the two sub-formulas of the formula having the introduced $\parr$ as main connective) it has to be an identity-axiom, as expected. ⊠

Lemma 26 Let N be a mono-conclusion cut-free proof-net. If the terminal link of N is a $\parr$-link whose premises (up to associativity) all are conclusions of axioms, then N is a $\parr$-ized identity-axiom.

Proof By lemmas 24 and 25. ⊠

Lemma 27 In any mono-conclusion proof-net, there exists a $\parr$-link.

Proof (induction on sequentialization) Let $\mathsf{N} : A$ a proof-net. Let π a sequentialization of N and r its last rule:

$$\pi \left\{ \frac{\vdots}{\vdash A}\, r \right.$$

1. The case $r = \mathsf{axm}$ is impossible.
2. If $r = \parr$-rule, we have it.
3. If $r = \otimes$-rule, then $A{:=}A'{\otimes}A''$ and $\pi := \left\{ \dfrac{\begin{matrix} \vdots\, \pi_1 & \vdots\, \pi_2 \\ \vdash A' & \vdash A'' \end{matrix}}{\vdash A' \otimes A''}\, r \right.$. By induction hypothesis on π_i, one concludes. ⊠

Lemma 28 Let N a mono-conclusion proof-net. If no splitting $\parr$ has a $\otimes$ among its premises (up to associativity and commutativity), then N is in canonical form (tensorization of $\parr$-ized axioms).

Proof Let us consider the terminal vertex of N.

1. If it is a $\parr$-link, being terminal, it is a splitting one. Thus by our main hypothesis each of its premises (up to associativity) is conclusion of an axiom. So, by lemma 26, N is a $\parr$-ized axiom.
2. If it is a $\otimes$ node, the mono-conclusion proof-net N has no terminal $\parr$. So, lemma 21 applies, and deleting that $\otimes$, one gets proof-nets which themselves are mono-conclusion (else, N itself would not be mono-conclusion). While one of the mono-conclusion proof-nets so produced ends with a $\otimes$-link, we are allowed to use lemma 21 again. Obstinately performing the corresponding $\otimes$ deletion process until it ends, we eventually get a set of mono-conclusion proof-nets M_i, each of them ending with a $\parr$-link (indeed, any such M_i

being mono-conclusion, the case where its conclusion vertex is conclusion of an identity axiom link is excluded). Those $\parr$-links are themselves splitting (in N) and, by our main hypothesis, they are thus surmounted by identity-axioms only. Hence by lemma 26, they are the $\parr$-links of $\parr$-ised axioms. So that N actually is a k-ary tensor ($k \geq 1$) of $\parr$-ized axioms. ⊠

Notation

- N^- is the proof-net one gets from N by deleting iteratively and obstinately terminal $\parr$-links (any conclusion vertex in N^- thus is conclusion of either a $\otimes$-link or an identity-axiom link).
- let v be the conclusion vertex of a given 'terminal' (up to associativity) $\parr$-link in N. Then $\mathsf{N}_{\uparrow}^{v-}$ is the proof-net obtained from N by deleting iteratively terminal $\parr$-links only up to v (i.e. we delete only those terminal $\parr$-links that one can reach without deleting v).

Lemma 29 Let N be a mono-conclusion cut-free proof-net whose terminal link is a $\parr$-link. If N includes a $\otimes$ link, then N contains a $\overset{\otimes}{\parr}$ for which $\overset{\mathrm{wp}}{\rightsquigarrow}^{-}$ rewriting is correct.

Proof Every terminal vertex of N^- is conclusion of an axiom or a $\otimes$-link and, by the $\otimes$ existence hypothesis and lemma 26 applied to N, at least one of those vertices is conclusion of a $\otimes$-link. Hence, by lemma 21 applied to N^-, at least one of them is a splitting $\otimes$. Let v_1 be the conclusion vertex of that $\otimes$. Because N is mono-conclusion and terminates with a $\parr$-link, in N, v_1 is premiss of a $\parr$-link. Let v_2 the conclusion vertex of that $\parr$-link. Up to $\parr$ associativity and commutativity, the other premiss v_3 of that $\parr$-link can be chosen terminal in N^-. So that in $\mathsf{N}_{\uparrow}^{v_2-}$, the superposition $\overset{v_1}{v_2}$ (or more precisely the superposition of the corresponding links) forms a double splitting $\overset{\otimes}{\parr}$. By proposition 23, performing in $\mathsf{N}_{\uparrow}^{v_2-}$ the corresponding $\overset{\mathrm{wp}}{\rightsquigarrow}^{-}$ reduction step preserves correctness. If we finally reintroduce as many terminal $\parr$-links as previously dropped from N (an operation which always preserves correctness), one finally gets a correct $\overset{\mathrm{wp}}{\rightsquigarrow}^{-}$ reduct of N.

⊠

Remark 3 Let N be a cut-free proof-net. If N is mono-conclusion, then any proof-net N_+ (see def. 20) produced by a splitting of N due to a splitting $\parr$ in N, has no other conclusions than the two premises of that splitting $\parr$.

Theorem 30 Let N a mono-conclusion cut-free proof-net. Then either N is in canonical form (n-ary $\otimes$ of $\parr$-ized axioms) or there exists in N a $\overset{\otimes}{\parr}$ whose $\overset{\text{WD}}{\rightsquigarrow}^{-}$ rewriting is correct.

Proof N is mono-conclusion. So by lemma 27, it contains a $\parr$-link. Hence, by lemma 22, there exists a splitting $\parr$. Thus:

1. If none of those splitting $\parr$ is surmounted (up to $\parr$-associativity) by a $\otimes$-link: then, by lemma 28, N is a n-ary tensor of "$\parr$-ized axioms".
2. Else, there exists a splitting $\parr$ surmounted (up to $\parr$-associativity) by a $\otimes$-link: let us complete N^{+} (the "upper" proof-net produced by the corresponding splitting) by applying a $\parr$-link to both its conclusions (see remark 3). Applying lemma 29 to that proof-net, one can find above its terminal $\parr$-link, a $\otimes$ s.t. the corresponding $\overset{\otimes}{\parr}$ is doubly splitting (in that proof net). The $\overset{\text{WD}}{\rightsquigarrow}^{-}$ step thus preserves correctness (and remains so when performed in the original proof-net: indeed, proof-nets are closed by the replacement of a mono-conclusion sub-proof-net by a mono-conclusion proof-net). ⊠

Remains now to check that the $\overset{\text{WD}}{\rightsquigarrow}^{-}$ rewriting is noetherian. For this, we will use the definition below, inspired by the one given in [4] for $\overset{\text{WD}}{\rightsquigarrow}$. Let us first present two notations. Omitting in a given cut-free structure the identity-axioms, one gets a multi-set of trees which, following ordinary conventions for spatial representation of partial orders, defines a partial strict order $<$ over the vertices of the structure (so $<$ means "strictly below"). Also, in what follows, $\wedge$ stands for the 'infimum' relative to $<$.

Definition 31 The complexity $\mathsf{C}(\mathsf{v})$ of a vertex v in N is defined by

$$\mathsf{C}(\mathsf{v}) = \sharp\{\mathsf{p} \text{ a } \parr\text{-link in } \mathsf{N},\ \text{s.t. } \mathsf{p}\wedge\mathsf{v} \text{ is a } \parr\text{-link s.t. } \mathsf{p}\wedge\mathsf{v} < \mathsf{v}\}$$

Remark If N is a mono-conclusion proof-net, then for any v in N, $\mathsf{C}(\mathsf{v})$ is the number of $\parr$-links in N which are not above v.

Definition 32 The complexity $\mathsf{C}(\mathsf{N})$ of a proof-net N is defined as:

$$\sum_{\mathsf{t}\ \otimes\text{-link in } \mathsf{N}} \mathsf{C}(\mathsf{t})$$

Proposition 33 The $\overset{\text{WD}}{\rightsquigarrow}_{1}^{-}$ rewriting rule makes the complexity decrease. Associativity and commutativity, however, left it unchanged

Proof Immediate adaptation of the proof given in [4] by exchanging the rôle of $\otimes$ and $\parr$ in it.

Theorem 34 *(Completeness)* Any mono-conclusion proof-net can be obtained from a proof-net in canonical form (a big $\otimes$ of $\wp$-ized axioms links) by $\overset{\text{WD}}{\rightsquigarrow}$ rewriting.

Proof By theorem 30 and proposition 33.

4 Epilogue

This result was first proved to build a bridge between MLL proof-nets and 'deep inference' formalisms ('Calculus of Structures' for multiplicative linear logic) introduced by A. Guglielmi [7] and others.

Roughly speaking, a given derivation in the multiplicative fragment of the calculus of structures appears to be but a notation for a given $\overset{\text{WD}}{\rightsquigarrow}$ rewriting strategy. Roughly only, notably because in deep inference systems 'generators' are only $\wp$-ized axioms (not n-ary $\otimes$ of them), a specific construction being separately added to introduce new $\wp$-ized axioms *at any time of the rewriting.*

References

[1] Girard J.-Y. (1987), Linear logic. *Theoretical Computer Science*, 50:1–102.

[2] Danos V. and Regnier L. (1995), The structure of multiplicatives. *Archives for Mathematical Logic*, 28, 181-203, 1989.

[3] Danos V. (1990), La logique linéaire appliquée à l'étude de divers processus de normalisation (principalement du λ-calcul). *Thèse de doctorat*, Université Paris 7, juin 1990.

[4] Maieli R. and Puite Q. (2005), Modularity of proof nets: generating the type of a module. *Archive for Mathematical Logic*, Volume 44, Number 2, 167-193, February 2005.

[5] Danos V., Joinet J-B., Schellinx H. (2003), Computational isomorphisms in classical logic. *Theoretical Computer Science*, Vol. 294, issue 3, pp.353-378", 2003

[6] Devarajan H., Hughes D., Plotkin G., Pratt V. (1999), Full completeness of the multiplicative linear logic of Chu spaces. *Logic in Computer Science*, 234-242, 1999

[7] Guglielmi A. (2004), A System of Interaction and Structure. *Technical Report WV-02-10* (8 November 2004), International Center for Computational Logic Technische Universität Dresden, 01062 Dresden, Germany to appear on *ACM Transactions on Computational Logic*

[8] Bechet D., P. de Groote, Retoré C. (1997), A complete axiomatisation for the inclusion of series-parallel orders. *RTA 97* LNCS volume 1232, 1997

[9] Retoré C. (1999), Handsome proof-nets: R&B graphs, perfect matchings and Series-Parallel graphs. *INRIA Research report RR-36-52*, 1999