Strong Normalization for All-Style $\mathbf{LK}^{tq}$

Extended Abstract

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Abstract. We prove strong normalization of *tq-reduction* for all standard versions of sequent calculus for classical and intuitionistic (second and first order) logic and give a perspicuous argument for the completeness of the *focusing* restriction on sequent derivations.

1 Introduction

In a recent paper ([3]) V. Danos and two of the authors gave an exhaustive analysis of the cut elimination procedure for Gentzen's sequent calculus for (first- and second order) classical logic **LK**, essentially by means of faithful interpretations ('decorations', cf. [7], [10], [4]) in linear logic: a classical proof π is mapped to a linear proof $D(\pi)$ in such a way that π is equal to its linear image's *skeleton* (i.e. the classical proof that, trivially, underlies a linear one). Consequently the *dynamics* (i.e. behaviour under cut-elimination) of $D(\pi)$ can be 'pulled back' to the original derivation π, which leads to the formulation of the so-called *tq-normalization* of (classical) second order sequent calculus.
The method and results appear to be of considerable theoretical, but also of practical, importance. A valuable observation e.g. is that several restrictions on the form of sequent proofs are *stable* under *tq*-normalization. This is of interest for the theory of automatic theorem proving, as it provides elegant proofs of *completeness* for the class of derivations satisfying these restrictions. In section 7 we will show how by this method one obtains elegant proofs of completeness for the so-called *focusing*-restriction ([1], [11]) on derivations in intuitionistic, classical and even linear sequent calculus.
The formulation of classical sequent calculus **LK** that we will consider in this paper is based upon the calculus obtained from the two sided sequent calculus for *linear logic* (see the appendix) by leaving out the exponential rules, and putting back in the usual structural rules of weakening and contraction known from calculi manipulating sequents $\Gamma \Rightarrow \Delta$, where $\Rightarrow$ is the *entailment sign* of

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the calculus, and Γ, Δ are finite *multisets* of formulas, i.e. sets with multiplicities, or (equivalently) lists modulo the order in which the entries are given.
Observe that this means that we have the choice between two *styles* ('additive', resp. 'multiplicative' style, terminology that stems from linear logic) for the introduction rules of the binary connectives. If, for a given connective, the introduction rules for *both* sides are of the same style, we say the connective is *homo-style*. We will distinguish homo-style connectives by a sub- or superscript, a or m: $\overset{m}{\to}, \overset{a}{\to}, \wedge_m, \wedge_a, \vee_m, \vee_a$. The set of rules for classical sequent calculus with homo-style additive and multiplicative binary connectives is given as an appendix. However, one might choose for binary connectives that are *hetero-style*, i.e. having (an) additive (mutiplicative) unary introduction rule(s) and a(n) multiplicative (additive) binary introduction rule. Hetero-style binary connectives hence are determined by the pairs $\langle a, m \rangle$, $\langle m, a \rangle$ whose components indicate the style of the unary, resp. that of the binary introduction rule(s). Here are the rules for the two hetero-style implications:[4]

$$(\mathrm{R}\overset{am}{\to})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \overset{am}{\to} B, \Delta} \qquad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \overset{am}{\to} B, \Delta} \qquad (\mathrm{L}\overset{am}{\to})\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \overset{am}{\to} B \Rightarrow \Delta_1, \Delta_2}$$

$$(\mathrm{R}\overset{ma}{\to})\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \overset{ma}{\to} B, \Delta} \qquad (\mathrm{L}\overset{ma}{\to})\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \overset{ma}{\to} B \Rightarrow \Delta}$$

Hence for each binary connective there are four distinct sets of introduction rules, each of which, when used in the context of the usual classical sequent calculus with structural rules of weakening and contraction, is obviously complete for classical provability.

In order to overcome the two major sources of indeterminism in Gentzen's original procedure of cut-elimination for classical sequent calculus (cf. [3]) we introduce two more extensions, one of the language, the other of the calculus:
• Each formula comes equipped with a mapping of the set of its subformulas into a 'colour space' $\{t, q\}$. When necessary, we will make explicit the colour ϵ of the formula itself by means of a superscript: A^ϵ.
• The multiplicative unary rules come in two types, prescribing whether in the 'key-step' of cut-elimination corresponding to the connective, the cut on the left subformula comes before that on the right one, or conversely: we speak of the *orientation* of the unary multiplicative rules.
The system thus defined will be referred to as 'all-style' $\mathbf{LK}^{tq}$; it has complete sets of rules for all four possible types of each of the binary connectives.
Most of the versions of sequent calculus for classical logic known from the literature can be (more or less directly) obtained as *fragments* of this 'all-style' $\mathbf{LK}^{tq}$, by choosing one complete set of rules for each of the binary connectives and simply forgetting about colours and orientations.
The present paper contains in some detail a proof of *strong normalization* for 'all-style' $\mathbf{LK}^{tq}$, hence *a fortiori* for any of its (complete) fragments.

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[4] The implication $\overset{am}{\to}$ corresponds e.g. to the arrow in 'Free Deduction' ([8]).

We will assume that the reader has some acquaintance with linear logic, especially with *proof nets* and their reduction.

2 Some terminology: main-active interspaces

The following conventions are used in distinguishing between the occurrences of formulas in a given logical rule, e.g. L$\rightarrow$:

$$\frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \rightarrow B \Rightarrow \Delta_1, \Delta_2}$$

The formula $A \rightarrow B$ is called the *main* formula of the rule with main connective $\rightarrow$; the occurrences A and B in the premisses will be referred to as the *active* formulas; all other occurrences are said to be *passive* and together will be referred to as the *context*. In the special case of a cut, active formula occurrences are also termed cutformulas.
In case of an identity axiom we say that *both* formula-occurrences are main.
We will restrict the use of the term 'main' to logical rules and identity axioms. In case of structural rules, if necessary, we speak of the *weakened*, respectively *contracted* formula.

Given an occurrence of a formula A in a sequent derivation π, we will speak of "the tree of A's ancestors in π", meaning the tree-like structure obtained by tracing upwards in π all formula occurrences corresponding to our initial A, up to the introductions of A by axiom, logical or weakening rule (these form the tree's leaves). The reader who wishes to do so, will easily provide the (long and boring) inductive definition.

In an occurrence in a proof π of a rule having as conclusion a sequent σ and sequent(s) σ' (and σ'') as premise(s), σ is called the *successor-sequent* of σ' (and σ'') in π. Similarly we will speak of the successor in σ of a formula occurrence in σ', σ'' (using the terminology of [2]: the formula occurrence A in σ is the successor of the formula occurrence A in σ', σ'' iff both occurrences are in the same *identity class*).
A *block* (of sequents) in π is a sequence of sequents $\sigma_0, \sigma_1, \ldots, \sigma_n$ in π such that σ_{i+1} is a successor of σ_i. Note that a block of sequents unambiguously defines a block of (successive occurrences of) *rules* in π.
Given an occurrence of a formula A in some sequent σ in a proof π, such that A has just been introduced (is *'newly born'*, i.e., is either main, or has just been weakened), we inductively define a sequence of occurrences $A_0, \ldots, A_n$ of A, together with a block B_A of sequents $\sigma_0, \ldots, \sigma_n$, by
(1) $\sigma_0 := \sigma, A_0 := A$;
(2) if $\sigma_i \in \mathsf{B}_A$, A_i is not active, and σ_{i+1} is the successor of σ_i, then $\sigma_{i+1} \in \mathsf{B}_A$, and A_{i+1} is the successor of A_i in σ_{i+1};
(3) that's all.
Hence B_A will contain σ, σ's successor, et cetera, down to either π's concluding sequent, or a sequent in which A is *active*. We will refer to B_A as A's *main-active interspace* or *m-a-interspace*.

With each *active* occurrence of a formula A, we can associate the finite set $\{B_{A_1}, \ldots, B_{A_n}\}$ of m-a-interspaces, where each A_i corresponds to a leaf of A's tree of ancestors.

3 Cut-elimination: the *tq*-protocol

An occurrence of a coloured formula on the left-hand side (resp. right-hand side) of the entailment-sign in a sequent is said to be *attractive* if its colour is t (resp. q): the terminology is introduced to remind us that the subproof of the sequent containing the non-attractive active cutformula "has to move first". We will often use the following alternative iconic notation for A^t (resp. A^q), namely $\overrightarrow{A}$ (resp. $\overleftarrow{A}$).
In each instance of a cut rule in an $\mathbf{LK}^{tq}$-proof, the cutformula will be coloured either t or q. Thus, using our iconic notation, cuts are of one of the two following forms[5]:

$$\frac{\Rightarrow \overrightarrow{A} \qquad \overrightarrow{A} \Rightarrow}{\Rightarrow} \qquad \text{and} \qquad \frac{\Rightarrow \overleftarrow{A} \qquad \overleftarrow{A} \Rightarrow}{\Rightarrow}$$

Let us call the subderivation containing the *attr*active occurrence of the cutformula the *attr*acting subderivation.

Definition 1 (tq-protocol) *Reduction according to tq-protocol proceeds via two possible types of steps, 'structural' ones, S1 and S2, and 'logical' ones, L ('key-steps'):*

— An L-step (to be specified for each of the logical connectives) applies when both cutformulas are main in a logical rule. (We obtain as descendants one or two cuts on the immediate subformula(s) of the cutformula. In case of two descendants, the order in which these cuts are applied is determined by the orientation of the unary rule.)

— In case no L-step is applicable, necessarily an S-step applies, which consists in 'transporting' one of the cut's subderivations up the tree of the cutformula's ancestors in the other one, duplicating it and contracting the context whenever passing an instance of contraction (or via the context of a binary additive rule); this process ends when reaching instances of introduction in an axiom, in which case the resulting 'axiom-cuts' are reduced immediately, *when reaching an introduction by weakening, which are replaced by weakenings of the context formulas, or when reaching instances of introduction of the main connective of the cutformula.*

Of course, now one needs to know which *of the two subderivations has to move. This is decided by asking whether or not the attractive cutformula is main* in a logical rule. *If the answer is* "yes!", *we transport the attracting subderivation (S2); if it is* "no!", *we transport the other one (S1).*
And that's it.

[5] In depicting proof figures, we will only indicate the context when this is relevant to the argument.

Note that neither the choice of orientations nor that of colours has to do with imposing a reduction *strategy*. We do not select redexes, but rather the way we reduce them.

4 Stability

A derivation in sequent calculus is a tree, in which, in the unique branch downward from some leaf (an *axiom*), we can follow occurrences of formulas from their introduction ('being main') to submergence ('being active'); in between in general many things will happen to *other* formula-occurrences. This 'meanwhile' is contained within what we have called the *main-active interspace* of the occurrence. If *nothing* happens in between, we say that the m-a-interspace is *flat*: the formula is 'born', and immediately put to work.
Let π' denote a one-step tq-reduct of π. Note that, trivially, *any* occurrence of a logical rule r in π' in which some occurrence of a formula, say A, is active, stems from a *unique* occurrence r_* of (the same) logical rule in π in which an occurrence A_* of (the same up to substitution) formula is active (we call r_*, resp. A_*, the *lift* of the occurrence r, resp. A). Conversely, we define the set of *residues* of a given occurrence s of a logical rule in π as the set of all occurrences r of logical rules in π' such that $r_* = s$.
Similarly, any occurrence of a cut c in π' with cutformula A stems from a unique occurrence c_* in π with cutformula A_*, except for those occurrences of cut that are created in an L-step (which are said to have *no* lift in π, and the reduced logical cut does *not* have a residue in π', cf. [3]).
As a consequence, given an *m-a*-interspace $B_A = \sigma_0, \dots, \sigma_n$ in π, such that σ_n's successor a is a logical or a cut rule, we can define the set of residues of B_A in π' as the set of all *m-a*-interspaces in π' associated with the corresponding active formulas in all residues of a in π'.
The following lemma says that when B_A is *flat*, under certain conditions the same will hold for all of B_A's residues.

Lemma 2 (stability) *Let π be an* $\mathbf{LK}^{tq}$*-proof, π' a one-step tq-reduct of π, and B_A a flat m-a-interspace of some formula occurrence A. Suppose A is attractive or m is a logical rule or a weakening. Then all residues of B_A in π' are flat.*

proof: Consider the schematic form of such an m-a-interspace in π:

$$\dfrac{\dfrac{\Delta}{\Delta', A}\,\mathrm{m} \qquad \Sigma}{\Delta', \Sigma'}\,\mathrm{a}$$

where 'm' denotes the rule that introduces A, 'a' the rule where A is active.
Reducing a cut in π obviously can only affect B_A in case $\Delta' \cup \{A\}$ contains (an ancestor of) the cutformula. Suppose A itself is the cutformula (hence 'a' is a cut). If the reduction step is an L-step, then B_A has no residue in π', and our claim vacuously holds.

If π_1 ends in a weakening on A or is an axiom $A \Rightarrow A$ and the cutformula is attractive, again there is no residue; similarly there is no residue in case π_1 ends in a weakening and the cut is of type S2. In all other cases π_1 will be the transported subderivation, independent of A's colour. For the leaves of A's tree of ancestors in π_2 corresponding to introductions in an axiom or a weakening, there are no residues of B_A; all residues will correspond to the introductions of ancestors of the cutformula in π_2 in a logical rule, and all of these are flat.
In case it is Δ' that contains an ancestor of the cutformula: either a subproof of π containing B_A will be transported, resulting in $m \geq 0$ flat residues; or the cut will 'just pass through' B_A, not affecting its flatness - unless π_1 is an identity-axiom, but then, by hypothesis, the occurrence of A that concerns us is attractive, therefore the cutformula is non-attractive. Hence the substitution will necessarily replace the axiom by a derivation having a last rule in which A is main. ⊠

Let us say that an m-a-interspace B_A is of type m (resp. w, ax) if m is a logical rule (resp. a weakening, an identity axiom). Then inspection of the proof shows that in fact:
- if B_A is flat of type w, all its residues are flat of type w;
- if B_A is flat of type m (corresponding to rule r), all its residues are flat of type m (corresponding to the same rule r);
- if A is attractive and B_A is flat of type ax, its residues are flat of type ax or of type m.
Ordering the three types of cut in the obvious way, by $S1 > S2 > L$, a simple corollary of the stability-lemma is that tq-reduction never increases a cut's type:

Proposition 3 *If c is a cut in π, and π tq-reduces to π', then (i) if c is a logical cut, all of c's residues in π' are logical cuts; (ii) if c is of type S2, none of c's residues in π' is of type S1.* ⊠

5 Strong normalization for 'all-style' $\mathbf{LK}^{tq}$

We will show that all tq-reduction sequences starting from a given derivation in the 'all-style' classical (second order) calculus are *finite*. The proof consists in a reduction to the well known strong normalization property for second order multiplicative *proof nets*: we will construct 'linear decorations', proof-by-proof embeddings into *taLL proof nets* (the multiplicative fragment of the system $\mathbf{PN}_2$ introduced in [5], and the treatment of quantifiers as in [6], extended with *tamed additives*, where additive boxes are such that auxiliary doors always are of the form $?X$, which enables a *simulation* of these (limited, but sufficient for our purposes) additives by means of multiplicatives, see [3]). These embeddings are homomorphisms with respect to the normalization: starting from the proof net $D^-(\pi)$ that we will associate with a given $\mathbf{LK}^{tq}$-derivation π, we can *simulate* π's tq-normalization; i.e., for any tq-reduction sequence $\pi \rightarrow \pi_1 \rightarrow \ldots \rightarrow \pi_k \rightarrow \ldots$ we have a proof net reduction sequence $D^-(\pi) \twoheadrightarrow D^-(\pi_1) \twoheadrightarrow \ldots \twoheadrightarrow D^-(\pi_k) \twoheadrightarrow \ldots$ where each tq-reduction step corresponds to one or more proof net reduction steps, except, possibly, in case the tq-step is an L-step on formulas whose main connective is a ***negation*** (this is because in proof nets negation is a ***defined*** notion); however it is easy to see that in a given tq-reduction sequence there can only occur ***finite*** sequences of such ***invisible*** steps. Therefore an ***infinite tq***-reduction will always induce an infinite proof net reduction, thus contradicting the strong normalization property of proof net reductions.

5.1 The inductive decoration of homo-style sequent proofs

We inductively associate to a ***homo-style*** $\mathbf{LK}^{tq}$-proof π of $\Gamma^t, \Lambda^q \Rightarrow \Delta^t, \Sigma^q$ a taLL proof net $D^-(\pi)$ with conclusions $(!?D(\Gamma))^\perp, (!D(\Lambda))^\perp, ?D(\Delta), ?!D(\Sigma)$, where $D(.)$ stands for some, as yet unspecified, ***modal translation*** of $\mathbf{LK}^{tq}$-formulas ϕ to linear formulas $D(\phi)$, later to be defined by induction on the complexity of such formulas. (For notational convenience in fact we will represent this proof net by a linear ***sequent proof***, an element of the equivalence class of sequent proofs corresponding to the net.) The modalities $!?, !, ?$ and $?!$ prefixing the D-translated formulas are called the *context-* or *surface*-modalities.
Here is the general pattern of what we call an *inductive decoration*: the axioms $A^t \Rightarrow A^t$ and $A^q \Rightarrow A^q$ become respectively

$$\frac{?D(A) \Rightarrow\ ?D(A)}{!?D(A) \Rightarrow\ ?D(A)} \qquad\qquad \frac{!D(A) \Rightarrow\ !D(A)}{!D(A) \Rightarrow\ ?!D(A)}$$

Given (a) decorated proof(s) of $!?D(\Gamma_i), !D(\Lambda_i) \Rightarrow ?D(\Delta_i), ?!D(\Sigma_i)$ (abbreviated as $\mu D(.)$), let us suppose that we continue by applying a *logical rule*. We then proceed as in

$$\cfrac{\cfrac{\cfrac{\cfrac{\mu D(\text{premises})}{\text{pre-modalization of active formula(s)}}}{\text{linear version of the rule}}}{\text{post-modalization of the main formula}}}{\mu D(\text{conclusion})}$$

(By pre- and post-modalization we mean: adding exponentials to a formula *before*, resp. *after* its being active in some rule.)
If the proof continues with a *structural rule*, we proceed similarly, but now there are *no* pre- and *no* post-modalization steps.
If, finally, it continues with a *cut rule*, then we distinguish the three possible types of cut.
(1) If the cut is logical, both occurrences of the cutformula A have just been main in a logical rule. We then *remove* their respective post-modalizations, and cut directly on $D(A)$: in the decoration of an L-cut, the cutformula is *without* surface-modality.
(2) If the cut is of type S2, then the attractive occurrence of the cutformula A has just been main in a logical rule. We *remove* the second post-modalization

(dereliction), and cut on $!D(A)$, resp. $?D(A)$: in the decoration of an S2-cut the cutformula has a surface-modality ***of length 1***.
(3) If the cut is of type S1, we add a post-modalization (promotion) to the decoration of the premise that contains the non-attractive occurrence of the cutformula, hence cut on $!?D(A)$, resp. $?!D(A)$: in the decoration of an S1-cut the cutformula has a surface-modality ***of length 2***.
(Observe how the length of surface-modalities reflects the natural order of the types of cuts.)
If we ask that the number of exponentials used in the modal translation D be as small as possible (so e.g. $D(p) := p$ for atoms p), the above procedure of decorating the logical introduction rules almost completely determines D. Its definition is a simple matter of unifying the possible minimal decorations of the active formulas in the unary and binary introduction rules:

$A \stackrel{m}{\rightarrow} B$

	t	q
t	$!?D(A) \multimap !?D(B)$	$!?D(A) \multimap ?!D(B)$
q	$?!D(A) \multimap ?!?D(B)$ $!?!D(A) \multimap !?D(B)$	$?!D(A) \multimap ?!D(B)$

$A \stackrel{a}{\rightarrow} B$

	t	q
t	$!?D(A) \leadsto !?D(B)$	$!?D(A) \leadsto ?!D(B)$
q	$?!D(A) \leadsto !?D(B)$	$?!D(A) \leadsto ?!D(B)$

	$\forall X A$
t	$\forall X !?D(A)$
q	$\forall X ?!D(A)$

	$\neg A$
t	$(!?D(A))^{\perp}$
q	$(?!D(A))^{\perp}$

In the (q,t) case of $\stackrel{m}{\rightarrow}$ we find *two* distinct minimal solutions. The first one, $?!D(A) \multimap ?!?D(B)$, corresponds to the orientation "A-cut under the B-cut" during the implication logical step; the second one, $!?!D(A) \multimap !?D(B)$, to the opposite orientation. (In the other cases the distinction is not relevant, hence doesn't appear in the decoration: there the two cuts commute. Also, we can (and we will), without loss of generality, limit ourselves to the case of implications, universal quantification and negation, as the D-translation of the other connectives and quantifier follow by a simple 'duality correspondence'. See [3] for a more detailed discussion.)
In subsections 5.2 and 5.3 we will show:

Theorem 4 (simulation theorem) *The mapping D^- of homo-style* $\mathbf{LK}^{tq}$*-proofs to taLL proof nets is a homomorphism with respect to normalization.*

5.2 The simulation of structural reduction steps

To start the proof of theorem 4, let us consider a structural cut c in π.
- If c is a cut on A of type S1, then in the decoration of c in $D^-(\pi)$ the cutformulas have a surface-modality of length 2 and the decoration of the premise containing the ***non-attractive*** occurrence of the cutformula ends with a promotion: in $D^-(\pi)$ this subderivation is an ***exponential box*** with, say, $\mu D(A)$ as principal door, and the decorated cut is an exponential cut. Reduction of this cut in $D^-(\pi)$ means (possibly) duplicating and/or erasing the box, then cutting it on the derelictions

introducing the outermost exponential of ancestors of $(\mu D(A))^{\perp}$. These derelictions follow introductions in either an axiom or a logical rule. In case of an axiom we proceed by a promotion-dereliction reduction followed by a substition; in case of a logical rule we proceed by a promotion-dereliction reduction, thus obtaining the decoration of a cut of type S2. If in fact the non-attractive occurrence of the cutformula in π was main in a logical rule, we perform one more promotion-dereliction reduction, and find the decoration of a cut of type L. This corresponds precisely to the performance of the S1-step in π.

- If c is a cut on A of type S2, then in the decoration of c in $D^{-}(\pi)$ it is the decoration of the premise containing the *attractive* occurrence of the cutformula that ends with a promotion: in $D^{-}(\pi)$ this subderivation is an *exponential box* with, say, $\nu D(A)$ as principal door. Again the decorated cut is an exponential cut and reduction of this cut in $D^{-}(\pi)$ means (possibly) duplicating and/or erasing the box, then cutting it on the derelictions introducing the outermost exponential of ancestors of $(\nu D(A))^{\perp}$. These derelictions follow introductions in either an axiom or a logical rule. In case of an axiom we proceed by a substition; in case of a logical rule we proceed by a promotion-dereliction reduction, thus obtaining the decoration of a cut of type L. This corresponds precisely to the performance of the S2-step in π.

- We now have obtained a proof net $(D^{-}(\pi))'$. By proposition 3 the type of residues of all *other* cuts in π did not increase. If the type of some cut did *decrease* we correspondingly lower the number of surface exponentials in $(D^{-}(\pi))'$ by promotion-dereliction reduction. Thus we obtain $D^{-}(\pi')$.

Hence we have shown: *If $\pi \longrightarrow_{tq} \pi'$ by a structural reduction step, then there is a non-void proof net reduction $D^{-}(\pi) \twoheadrightarrow D^{-}(\pi')$.*

5.3 The simulation of logical reduction steps

If the orientation of the unary multiplicative rule is "B-cut under the A-cut", the proof figure π on the left reduces to π' on the right:

$$\dfrac{\dfrac{\Rightarrow A \qquad B \Rightarrow}{A \stackrel{m}{\rightarrow} B \Rightarrow} \qquad \dfrac{A \Rightarrow B}{\Rightarrow A \stackrel{m}{\rightarrow} B}}{} \qquad \longrightarrow \qquad \dfrac{\dfrac{\Rightarrow A \qquad A \Rightarrow B}{\Rightarrow B} \qquad\qquad B \Rightarrow}{}$$

If the orientation is the other way round, of course π' changes accordingly.

We leave it to the reader to verify that in case the colouring is different from $A^{q} \stackrel{m}{\rightarrow} B^{t}$, $D^{-}(\pi)$ reduces by a logical reduction followed (possibly) by a commutative reduction step to a net which does not depend on the orientation of the $\mathbf{LK}^{tq}$-rule, and which reduces to $D^{-}(\pi')$ by zero or more promotion-dereliction reductions, depending on the type of the two cuts obtained. Here we will just consider the case $A^{q} \stackrel{m}{\rightarrow} B^{t}$.

In case the orientation is "A-cut under the B-cut", we have a $D^{-}(\pi)$ which by a logical reduction reduces to:

$$\dfrac{\Rightarrow ?!D(A) \qquad \dfrac{\dfrac{\dfrac{\dfrac{!D(A) \Rightarrow ?D(B)}{!D(A) \Rightarrow !?D(B)}}{!D(A) \Rightarrow ?!?D(B)}}{?!D(A) \Rightarrow ?!?D(B)}}{}}{\dfrac{\Rightarrow ?!?D(B) \qquad \dfrac{!?D(B) \Rightarrow}{?!?D(B) \Rightarrow}}{}}$$

and then by a commutative and a promotion/dereliction reduction becomes:

$$\dfrac{\Rightarrow ?!D(A) \qquad \dfrac{\dfrac{\dfrac{!D(A) \Rightarrow ?D(B)}{!D(A) \Rightarrow !?D(B)} \qquad !?D(B) \Rightarrow}{!D(A) \Rightarrow}}{?!D(A) \Rightarrow}}{}$$

which, by zero or more promotion/dereliction reductions, reduces to the net corresponding to the D^- of the "A-cut under the B-cut" reduction. The reader should verify that the other possible translation, $!?!D(A) \to !?D(B)$, indeed will give rise to the D^- of the "B-cut under the A-cut" reduction.

In the additive case we have the reduction

$$\dfrac{\dfrac{\Rightarrow B}{\Rightarrow A \overset{a}{\to} B} \qquad \dfrac{\Rightarrow A \qquad B \Rightarrow}{A \overset{a}{\to} B \Rightarrow}}{} \quad \longrightarrow \quad \dfrac{\Rightarrow B \qquad B \Rightarrow}{}$$

and similarly for the other unary introduction rule.

We leave the verification of the simulation to the reader, who is also invited to do the same in case of a logical reduction involving quantifiers. After having done so, she should be convinced of the following: *If $\pi \to_{tq} \pi'$ by a logical reduction step involving a quantifier or an implication, then there is a non-void proof net reduction $D^-(\pi) \twoheadrightarrow D^-(\pi')$.*

In the case of the reduction of a logical cut on a negated formula, say of $\neg A^q$, we have a reduction π to π' as in:

$$\dfrac{\dfrac{\Rightarrow ?!D(A)}{(?!D(A))^\perp \Rightarrow} \qquad \dfrac{\dfrac{!D(A) \Rightarrow}{?!D(A) \Rightarrow}}{\Rightarrow (?!D(A))^\perp}}{} \quad \longrightarrow \quad \dfrac{\Rightarrow ?!D(A) \qquad \dfrac{!D(A) \Rightarrow}{?!D(A) \Rightarrow}}{}$$

As *proof nets* these two figures are the same: the operation is *void*. However, in order to obtain $D^-(\pi')$ it might be necessary to proceed by performing one or two promotion/dereliction reductions; hence the simulation will be 'empty' iff the created cut in π' is structural of type S1. Anyway, as we already observed, one can only perform a finite number of such 'invisible' reductions in a row.

We conclude: *If $\pi \to_{tq} \pi'$ by a logical reduction step involving a negation, then there is a proof net reduction $D^-(\pi) \twoheadrightarrow D^-(\pi')$. However it might be the case that $D^-(\pi) = D^-(\pi')$.*

This finishes the proof of our simulation theorem for the tq-reduction of homostyle $\mathbf{LK}^{tq}$.

It enables us to state:

Theorem 5 (homo SN) *The tq-protocol applied to homo-style* $\mathbf{LK}^{tq}$ *is strongly normalizing.*

5.4 The inductive decoration of hetero-style sequent proofs

Both the multiplicative implication $\stackrel{m}{\to}$ and the additive implication $\stackrel{a}{\to}$ have a linear 'correspondent': $\multimap$, resp. $\leadsto$. This is of course not the case for the hetero-style implication $\stackrel{am}{\to}$ and $\stackrel{ma}{\to}$: linear logic is homo-style. We hence have to follow slightly more devious paths in order to show that tq- reduction of $\mathbf{LK}^{tq}$-derivations that contain them, is noetherian. But obviously in one way or another, we will have to interpret the hetero-style connectives by homo-style ones. Now, it is well known that, using the structural rules of weakening and contraction, all four formulations that we gave for the introduction rules of the binary connectives are *interderivable*, and all styles are consequently equivalent for provability. Our aim, however, lies beyond mere provability: it is the behaviour of proofs under (tq-)reduction that interests us, hence in following the (obvious) idea of using the derivability of hetero-style rules in the homo-style calculus, we have to make sure that these derived rules in home-style $\mathbf{LK}^{tq}$ simulate the intended dynamics of the hetero-style connectives: if s denotes a homo-style version of a hetero-style rule r, and $\pi[s/r]$ the homo-style proof obtained by replacing all hetero-style rules by their homo-style variants, then we need that, whenever a hetero-style proof π tq-reduces to π', there is a tq-reduction from $\pi[s/r]$ to $\pi'[s/r]$.
It turns out that one can guarantee this (literal) 'homo'-simulation property *only* in case the homo-style version of the hetero-style rule is such that the unavoidable structural rules to be added appear *before* the homo-version of the logical rule. In general, this is impossible.
We will therefore exploit the possibilities provided by linear logic to simulate dynamic subtleties by 'exponential boxing' of specific subderivations, and define (i) an extension of our translation D into linear logic to the hetero-style connectives and (ii) a way of decorating the hetero-style rules such that, as in the homo-style case, given a hetero-style proof π, the resulting proof net $D^-(\pi)$ simulates π's tq-reduction.

Let us first state the logical reductions corresponding to the two hetero-style implications: if the orientation of the unary multiplicative rule is "A-cut under the B-cut", then the proof figure π:

$$\dfrac{\dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \stackrel{ma}{\to} B \Rightarrow \Delta} \qquad \dfrac{A \Rightarrow B}{\Rightarrow A \stackrel{ma}{\to} B}}{\Gamma \Rightarrow \Delta}$$

reduces to π':

$$\dfrac{\dfrac{\Gamma \Rightarrow A, \Delta \qquad \dfrac{A \Rightarrow B \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \Rightarrow \Delta}}{\Gamma, \Gamma \Rightarrow \Delta, \Delta}}{\Gamma \Rightarrow \Delta}$$

If the orientation is the other way round, of course π' changes accordingly.
For $\overset{am}{\to}$, we have the reduction:

$$\dfrac{\dfrac{A \Rightarrow}{\Rightarrow A \overset{am}{\to} B} \qquad \dfrac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \overset{am}{\to} B \Rightarrow \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad \longrightarrow \quad \dfrac{\dfrac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad A \Rightarrow}{\Gamma_1 \Rightarrow \Delta_1}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Similarly for the other unary introduction rule.
Each of the two hetero-style connectives can be interpreted in two 'homo'-ways: either multiplicatively, or additively. Let us start with the first interpretation.

The multiplicative decoration

The additive binary rule

$$\dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \overset{ma}{\to} B \Rightarrow \Delta} \quad \text{becomes} \quad \dfrac{\dfrac{\Gamma \Rightarrow A, \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, \Gamma, A \overset{ma}{\to} B \Rightarrow \Delta, \Delta}}{\Gamma, A \overset{ma}{\to} B \Rightarrow \Delta}$$

The additive unary rules

$$\dfrac{A \Rightarrow}{\Rightarrow A \overset{am}{\to} B}\ ,\quad \dfrac{\Rightarrow B}{\Rightarrow A \overset{am}{\to} B} \quad \text{become} \quad \dfrac{\dfrac{A \Rightarrow}{A \Rightarrow B}}{\Rightarrow A \overset{am}{\to} B}\ ,\quad \dfrac{\dfrac{\Rightarrow B}{A \Rightarrow B}}{\Rightarrow A \overset{am}{\to} B}$$

Now let π be a proof containing occurrences of these heterostyle rules. Replace all instances by their multiplicative simulation, given above. We then extend the definition of inductive decoration, given in section 5.1, as follows: suppose we already have (a) decorated proof(s) with conclusion(s) $!?D(\Gamma_i), !D(A_i) \Rightarrow ?D(\Delta_i), ?!D(\Sigma_i)$ (abbreviated as $\mu D(.)$) and the original proof continues with a *logical rule having* $\overset{am}{\to}$ *or* $\overset{ma}{\to}$ *as main connective.* We then proceed as in

$\overset{ma}{\to}$

$$\begin{array}{c} \mu D(\text{premises}) \\ \hline \text{pre-mod. of active formulas} \\ \hline \text{linear version of the rule} \\ \hline \text{contraction of the context} \\ \hline \text{post-mod. of main formula} \\ \hline \mu D(\text{conclusion}) \end{array}$$

$\overset{am}{\to}$

$$\begin{array}{c} \mu D(\text{premises}) \\ \hline \text{pre-mod. of active formula(s)} \\ \hline \text{weakening on an active formula} \\ \hline \text{linear version of the rule} \\ \hline \text{post-mod. of main formula} \\ \hline \mu D(\text{conclusion}) \end{array}$$

$$D(A \overset{ma}{\to} B) := D(A \overset{m}{\to} B)$$

$D(A \overset{am}{\to} B)$

	t	q
t	$!?D(A) \multimap ?!?D(B)$	$!?D(A) \multimap ?!D(B)$
q	$!?!D(A) \multimap ?!?D(B)$	$!?!D(A) \multimap ?!D(B)$

The additive decoration

The multiplicative unary rule $\dfrac{A \Rightarrow B}{\Rightarrow A \overset{ma}{\to} B}$ becomes:

$$\begin{array}{c} A \Rightarrow B \\ \hline \Rightarrow A \overset{ma}{\to} B, B \\ \hline \Rightarrow A \overset{ma}{\to} B, A \overset{ma}{\to} B \\ \hline \Rightarrow A \overset{ma}{\to} B \end{array} \quad \text{or} \quad \begin{array}{c} A \Rightarrow B \\ \hline A \Rightarrow A \overset{ma}{\to} B \\ \hline \Rightarrow A \overset{ma}{\to} B, A \overset{ma}{\to} B \\ \hline \Rightarrow A \overset{ma}{\to} B \end{array}$$

The multiplicative binary rule $\dfrac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \overset{am}{\to} B \Rightarrow \Delta_1, \Delta_2}$ becomes:

$$\dfrac{\dfrac{\Gamma_1 \Rightarrow A, \Delta_1}{\Gamma_1, \Gamma_2 \Rightarrow A, \Delta_1, \Delta_2} \qquad \dfrac{\Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, B \Rightarrow \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2, A \overset{am}{\to} B \Rightarrow \Delta_1, \Delta_2}$$

Now let π be a proof containing occurrences of these heterostyle rules. Replace all instances by their additive simulation, given above. We then extend the definition of inductive decoration, given in section 5.1, as follows: suppose we already have (a) decorated proof(s) with conclusion(s) $!?D(\Gamma_i), !D(A_i) \Rightarrow ?D(\Delta_i), ?!D(\Sigma_i)$ (abbreviated as $\mu D(.)$) and the original proof continues with a *logical rule having* $\overset{am}{\to}$ *or* $\overset{ma}{\to}$ *as main connective*. We then proceed as in

$\overset{ma}{\to}$

$$\begin{array}{c} \mu D(\text{premises}) \\ \hline \text{pre-mod. of active formulas} \\ \hline \text{linear version(s) of the rule} \\ \hline \text{derelictions} \parallel \text{promotion} \\ \hline \text{contraction of main formulas} \\ \hline \text{post-mod. of the main formula} \\ \hline \mu D(\text{conclusion}) \end{array}$$

$\overset{am}{\to}$

$$\begin{array}{c} \mu D(\text{premises}) \\ \hline \text{pre-mod. of active formulas} \\ \hline \text{weakening of context} \\ \hline \text{linear version of the rule} \\ \hline \text{post-mod. of main formula} \\ \hline \mu D(\text{conclusion}) \end{array}$$

$D(A \overset{ma}{\to} B)$

	t	q
t	$?(!?D(A) \rightsquigarrow !?D(B))$	$?(!?D(A) \rightsquigarrow ?!D(B))$
q	$?(?!D(A) \rightsquigarrow ?!?D(B))$ $?(!?!D(A) \rightsquigarrow !?D(B))$	$?(?!D(A) \rightsquigarrow ?!D(B))$

$$D(A \overset{ma}{\to} B) := D(A \overset{m}{\to} B)$$

As an example, here are the 'multiplicative decoration' of the binary rule introducing $A^t \overset{ma}{\to} B^q$ and an 'additive decoration' of the unary rule introducing $A^q \overset{ma}{\to} B^t$ (for notational convenience we write X instead of $D(X)$):

$$\begin{array}{c} \dfrac{\dfrac{\mu\Gamma \Rightarrow ?A, \nu\Delta}{\mu\Gamma \Rightarrow !?A, \nu\Delta} \qquad \dfrac{\mu\Gamma, !B \Rightarrow \nu\Delta}{\mu\Gamma, ?!B \Rightarrow \nu\Delta}}{\mu\Gamma, \mu\Gamma, !?A \multimap ?!B \Rightarrow \nu\Delta, \nu\Delta} \\ \hline\hline \mu\Gamma, !?A \multimap ?!B \Rightarrow \nu\Delta \\ \hline \mu\Gamma, \mu'(!?A \multimap ?!B) \Rightarrow \nu\Delta \end{array}$$

$$\begin{array}{c} !A \Rightarrow ?B \\ \hline !A \Rightarrow !?B \\ \hline !A \Rightarrow ?!?B \\ \hline ?!A \Rightarrow ?!?B \\ \hline \Rightarrow ?!A \rightsquigarrow ?!?B, ?!?B \\ \hline \Rightarrow ?!A \rightsquigarrow ?!?B, ?!A \rightsquigarrow ?!?B \\ \hline \Rightarrow ?(?!A \rightsquigarrow ?!?B), ?!A \rightsquigarrow ?!?B \\ \hline \Rightarrow ?(?!A \rightsquigarrow ?!?B), ?(?!A \rightsquigarrow ?!?B) \\ \hline \Rightarrow ?(?!A \rightsquigarrow ?!?B) \\ \hline \Rightarrow \mu?(?!A \rightsquigarrow ?!?B) \end{array}$$

5.5 Simulation of the hetero-style reduction

As the ***structural reduction*** is completely determined by the surface-modalities, the simulation argument given in the homo-style case continues to hold for the hetero-style, yes indeed, in the ***all-style*** case. Note that, by definition of the inductive decoration, a decoration of a rule ***always*** ends by a post-modalization of the main formula; therefore a block corresponding to the decoration of a hetero-style rule can not be modified during a structural reduction step. Hence it suffices to verify that our decorations simulate the hetero-style logical reductions. Let's have a go at a multiplicatively decorated $A^q \xrightarrow{am} B^q$-reduction!
The decoration of the cut and its reduction then is as follows:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{!A \Rightarrow}{?!A \Rightarrow}}{!?!A \Rightarrow}}{!?!A \Rightarrow ?!B}}{\Rightarrow !?!A \multimap ?!B} \qquad \dfrac{\dfrac{\Gamma_1 \Rightarrow ?!A, \Delta_1}{\Gamma_1 \Rightarrow !?!A, \Delta_1} \qquad \dfrac{\Gamma_2, !B \Rightarrow \Delta_2}{\Gamma_2, ?!B \Rightarrow \Delta_2}}{\Gamma_1, \Gamma_2, !?!A \multimap ?!B \Rightarrow \Delta_1, \Delta_2}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \quad \twoheadrightarrow \quad \dfrac{\dfrac{\Gamma_1 \Rightarrow ?!A, \Delta_1 \qquad \dfrac{!A \Rightarrow}{?!A \Rightarrow}}{\Gamma_1 \Rightarrow \Delta_1}}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

which, by zero or more promotion/dereliction reductions, reduces to the net corresponding to the D^- of the reduct of the hetero-style cut. We leave the, straightforward, verification of the simulation in the other cases to the reader.

Theorem 6 (hetero SN) *The tq-protocol applied to hetero-style* $\mathbf{LK}^{tq}$ *is strongly normalizing.*

Both SN-theorems of course combine:

Corolary 7 (all-style SN) *The tq-protocol applied to all-style* $\mathbf{LK}^{tq}$ *is strongly normalizing.*

As one application of theorem 6 let us mention a proof of strong normalization for Parigot's system of Free Deduction **FD**, which uses *hetero-style* connectives, and the reduction of which is easily simulable in hetero-style $\mathbf{LK}^{tq}$ (cf. [12]).

5.6 Intuitionistic and linear tq-calculus

Intuitionistic sequent calculus **LJ** is obtained from classical sequent calculus **LK** by a uniform restriction on the form of sequents: one asks that for sequents $\Gamma \Rightarrow \Delta$ occurring in a derivation the succedent multiset Δ contains ***at most one*** element. Once due care is taken that this restriction is satisfied by the ***rules*** that constitute the calculus, all derivable sequents and all proofs are intuitionistically valid. Let $\mathbf{LJ}^{tq}$ be the calculus consisting in the intuitionistically restricted subset of the set of rules that constitute $\mathbf{LK}^{tq}$. As the intuitionistic restriction is obviously stable under tq-reduction, the results of the previous sections hold *a*

fortiori for $\mathbf{LJ}^{tq}$, which contains as complete subsystems many of the standard versions for intuitionistic sequent calculus.
In [9] it is sketched how, with similar results, the tq-analysis can be applied to sequent calculus for the modal logic **S4**. As the set of rules for *linear sequent calculus* itself in fact consists in a proper subset of that of **S4**, the above also applies to linear logic. In colouring formulas one has to ask that fomulas prefixed by $\Box$ or '!', ($\Diamond$ or '?') are coloured q (t). (We therefore call linear formulas occurring in the succedent (antecedent) that are prefixed by a '!' ('?') *attractively* modalized.) Observe that this means that one shows that tq-reduction of linear sequent calculus is noetherian by embedding linear logic into... linear logic... here's one *snake* that bites its own tail!

6 Focusing

We encountered at the very end of the previous section an example of a restriction on the form of sequent derivation that is stable under (tq-)normalization: the intuitionistic restriction, to sequents having at most one succedent formula. It is a stable restriction, but drastic, in the sense that when applying it to classical logic, the resulting system is no longer complete for (classical) provability. In this final section we will describe restrictions that are both *stable* under tq-reduction and *complete* for provability (in classical, intuitionistic, modal, linear logic).
In order to state these restrictions, recall the notions of *reversible* versus *irreversible rules* in classical and intuitionistic sequent calculus: binary additive rules, unary multiplicative rules, left existential and right universal quantifier rules, as well as *both* rules for negation, are reversible. All other rules are irreversible. In (modal and) linear sequent calculus the *dereliction* rules are irreversible as well.
Let us write $\mathbf{L}\mathcal{X}^{tq}$ to indicate any of the calculi $\mathbf{LK}^{tq}$, $\mathbf{LJ}^{tq}$, or $\mathbf{LL}^{tq}$.

Definition 8 *An* $\mathbf{L}\mathcal{X}^{tq}$*-derivation is called* η-restricted *iff all attractive formulas active in an irreversible or negation rule are* main; *it is called* focused *iff all non-atomic formulas that are either not or attractively exponentiated, and active in an irreversible or negation rule, are* main in a logical rule.

We call the sets of η-restricted $\mathbf{L}\mathcal{X}^{tq}$-derivations: $\mathbf{L}\mathcal{X}^{\eta}$; the sets of focused $\mathbf{L}\mathcal{X}^{tq}$-derivations: $\mathbf{L}\mathcal{X}^{\sharp}$.

Theorem 9 $\mathbf{L}\mathcal{X}^{\eta}$ *and* $\mathbf{L}\mathcal{X}^{\sharp}$ *are closed under tq-normalization.*

proof: This is a consequence of the stability lemma 2. Here's the detailed argument for *focused* proofs: suppose π is focused and tq-reduces to π'. Let A be a non-atomic, formula occurrence that is not or attractively exponentiated and active in an irreversible or a negation rule r in π'. Then r is a residue of an occurrence r_* of that same rule in π. Hence A_* is a non-atomic formula occurrence that is not or attractively exponentiated and active in an irreversible or negation rule in π; by induction hypothesis it is main in a logical rule: the unique main-active interspace associated with A_* is a flat block B_{A_*} in π. By stability all residues of B_{A_*} are flat; in particular B_A is flat, and A is main in the same logical rule in π'. ⊠

Theorem 10 (η-restriction is complete) $\mathbf{L}\mathcal{X}^\eta \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{L}\mathcal{X}^{tq} \vdash \Gamma \Rightarrow \Delta$.

proof: Let π be an $\mathbf{L}\mathcal{X}^{tq}$-proof of $\Gamma \Rightarrow \Delta$. We restrict π by adding cuts with little bits of canonical proofs of identities $X \Rightarrow X$ (so-called η-proofs, whence the name η-restriction, cf. [3]). For example, occurrences of irreversible implication left-rules

$$\frac{\Rightarrow A \qquad B \Rightarrow}{A \overset{m}{\multimap} B \Rightarrow} \quad \text{are replaced by} \quad \cfrac{\cfrac{\Rightarrow A \qquad \cfrac{A \Rightarrow A \qquad B \Rightarrow B}{A, A \overset{m}{\multimap} B \Rightarrow B}}{A \overset{m}{\multimap} B \Rightarrow B} \qquad B \Rightarrow}{A \overset{m}{\multimap} B \Rightarrow}$$

and occurrences of dereliction

$$\frac{\Rightarrow A}{\Rightarrow ?A} \quad \text{are replaced by} \quad \cfrac{\Rightarrow A \qquad \cfrac{A \Rightarrow A}{A \Rightarrow ?A}}{\Rightarrow ?A}$$

The reader is invited to find the replacements for the other irreversible rules. Call the resulting proof $\mathsf{str}(\pi)$. Now $\mathsf{str}(\pi)$ satisfies the η-constriction and $\mathsf{str}(\pi) \vdash \Gamma \Rightarrow \Delta$. ⊠

Note that, in case the derivation π we start from is normal, we obtain a normal proof $\mathsf{str}(\pi)'$ in $\mathbf{L}\mathcal{X}^{tq}$, simply by normalizing $\mathsf{str}(\pi)$. This is immediate from theorem 9. Hence the above shows us how to *transform* (normal) $\mathbf{L}\mathcal{X}^{tq}$-proofs into (normal) $\mathbf{L}\mathcal{X}^{\eta}$-proofs.[6] Observe that this transformation is a *projection*: if in the above one of the would-be constricted formulas is non-attractive or already main, eliminating the cut is an 'empty' operation.[7]

Theorem 11 (focusing is complete) $\mathbf{L}\mathcal{X}^\sharp \vdash \Gamma \Rightarrow \Delta$ *iff* $\mathbf{L}\mathcal{X}^{tq} \vdash \Gamma \Rightarrow \Delta$.

proof: Let π_0 be an $\mathbf{L}\mathcal{X}^{tq}$-derivation of $\Gamma \Rightarrow \Delta$. We *expand* π_0, in order to obtain a derivation π_1 in which all identity axioms are atomic. Let A be a non-atomic formula active in an irreversible or a negation rule; if the formula is not attractive and not exponentiated, then flip the colour of its identity class: if it was coloured t (q) it now is coloured q (t). This is possible because of the fact that all

[6] Which is the reason why sometimes we refer to the 'little cuts' introduced in this proof as 'constrictive morphisms'.

[7] Clearly, in order to guarantee this *projection-property*, it suffices to have *weak normalization*. In the first-order case there is a simple inductive argument that establishes tq-eliminability of cuts directly (cf. [3]; the main ingredient is proposition 3), without going through the trouble of simulating the reductions by means of proof nets. But of course this direct proof fails as soon as we consider the second-order extensions; whence we might as well go for the *strong* normalization right away.

identity axioms in π_1 are atomic! Now A is attractive. Repeat this operation for all non-atomic, non-attractive and non-exponentiated formulas that are active in irreversible or negation rules. This gives us an $\mathbf{L}\mathcal{X}^{tq}$-derivation π_2 of $\Gamma \Rightarrow \Delta$ (N.B.: colours have changed!) in which all non-atomic non-exponentiated formulas that are active in an irreversible or a negation rule are *attractive*. Then apply the transformation of the previous proof in order to get $\pi_3 := \mathsf{str}(\pi_2)' \in \mathbf{L}\mathcal{X}^{\eta}$, i.e. we branch the appropriate constrictive morphisms and eliminate them. The result is *focused*: it suffices to observe that having atomic identity axioms is stable under tq-reduction of the constrictive cuts in $\mathsf{str}(\pi_2)$. Finally, in π_3, we can put back the original colours. Hence $\mathbf{L}\mathcal{X}^{\natural} \vdash \Gamma \Rightarrow \Delta$. ⊠

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"merci, Vincent!'

Appendix

LK, classical logic

Identity axiom and cut rule: (Ax) $A \Rightarrow A$ $\quad$ (cut) $\dfrac{\Gamma_1 \Rightarrow \Delta_1, A \qquad A, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

Negation rules: (L$\neg$) $\dfrac{\Gamma \Rightarrow A, \Delta}{\Gamma, \neg A \Rightarrow \Delta}$ $\quad$ (R$\neg$) $\dfrac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow \neg A, \Delta}$

Multiplicative logical rules:

$$(\mathrm{L}\stackrel{m}{\to})\ \frac{\Gamma_1 \Rightarrow \Delta_1, A \qquad B, \Gamma_2 \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \stackrel{m}{\to} B \Rightarrow \Delta_1, \Delta_2} \qquad (\mathrm{R}\stackrel{m}{\to})\ \frac{\Gamma, A \Rightarrow B, \Delta}{\Gamma \Rightarrow A \stackrel{m}{\to} B, \Delta}$$

$$(\mathrm{L}\vee_m)\ \frac{\Gamma_1, A \Rightarrow \Delta_1 \qquad \Gamma_2, B \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2, A \vee_m B \Rightarrow \Delta_1, \Delta_2} \qquad (\mathrm{R}\vee_m)\ \frac{\Gamma \Rightarrow A, B, \Delta}{\Gamma \Rightarrow A \vee_m B, \Delta}$$

$$(\mathrm{R}\wedge_m)\ \frac{\Gamma_1 \Rightarrow A, \Delta_1 \qquad \Gamma_2 \Rightarrow B, \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge_m B, \Delta_1, \Delta_2} \qquad (\mathrm{L}\wedge_m)\ \frac{\Gamma, A, B \Rightarrow \Delta}{\Gamma, A \wedge_m B \Rightarrow \Delta}$$

Additive logical rules:

$$(\mathrm{R}\stackrel{a}{\to})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma \Rightarrow A \stackrel{a}{\to} B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \stackrel{a}{\to} B, \Delta} \qquad (\mathrm{L}\stackrel{a}{\to})\ \frac{\Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{\Gamma, A \stackrel{a}{\to} B \Rightarrow \Delta}$$

$$(\mathrm{R}\vee_a)\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow A \vee_a B, \Delta} \quad \frac{\Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \vee_a B, \Delta} \qquad (\mathrm{L}\vee_a)\ \frac{\Gamma, A \Rightarrow \Delta \qquad \Gamma, B \Rightarrow \Delta}{\Gamma, A \vee_a B \Rightarrow \Delta}$$

$$(\mathrm{L}\wedge_a)\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, A \wedge_a B \Rightarrow \Delta} \quad \frac{\Gamma, B \Rightarrow \Delta}{\Gamma, A \wedge_a B \Rightarrow \Delta} \qquad (\mathrm{R}\wedge_a)\ \frac{\Gamma \Rightarrow A, \Delta \qquad \Gamma \Rightarrow B, \Delta}{\Gamma \Rightarrow A \wedge_a B, \Delta}$$

Rules for the quantifiers (y, Y not free in Γ, Δ):

$$(\mathrm{L}\forall)\ \frac{\Gamma, A[t/x] \Rightarrow \Delta}{\Gamma, \forall x\, A \Rightarrow \Delta} \quad (\mathrm{R}\forall)\ \frac{\Gamma \Rightarrow A[y/x], \Delta}{\Gamma \Rightarrow \forall x\, A, \Delta} \quad (\mathrm{L}\forall_2)\ \frac{\Gamma, A[T/X] \Rightarrow \Delta}{\Gamma, \forall X\, A \Rightarrow \Delta} \quad (\mathrm{R}\forall_2)\ \frac{\Gamma \Rightarrow \Delta, A[Y/X]}{\Gamma \Rightarrow \Delta, \forall X\, A}$$

$$(\mathrm{L}\exists)\ \frac{\Gamma, A[y/x] \Rightarrow \Delta}{\Gamma, \exists x\, A \Rightarrow \Delta} \quad (\mathrm{R}\exists)\ \frac{\Gamma \Rightarrow A[t/x], \Delta}{\Gamma \Rightarrow \exists x\, A, \Delta} \quad (\mathrm{L}\exists_2)\ \frac{\Gamma, A[Y/X] \Rightarrow \Delta}{\Gamma, \exists X\, A \Rightarrow \Delta} \quad (\mathrm{R}\exists_2)\ \frac{\Gamma \Rightarrow \Delta, A[T/X]}{\Gamma \Rightarrow \Delta, \exists X\, A}$$

Structural rules: Weakening and contraction, left and right.

LL, classical linear logic

Identity axiom, cut, negation , quantifier rules: As in **LK**, but negation is written by means of a superscript ($A^{\perp}$). *Multiplicative and additive logical rules:* One obtains the linear logical introduction rules for the binary connectives by taking those for classical logic, and replacing $\wedge_m, \vee_m, \stackrel{m}{\to}, \wedge_a, \vee_a, \stackrel{a}{\to}$ by, resp. $\otimes, ⅋, \multimap, \&, \oplus, \rightsquigarrow$. There are *no* structural rules. But we have the
Exponential rules:

$$(\mathrm{W!})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \quad (\mathrm{W?})\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow ?A, \Delta} \quad (\mathrm{C!})\ \frac{\Gamma, !A, !A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta} \quad (\mathrm{C?})\ \frac{\Gamma \Rightarrow ?A, ?A, \Delta}{\Gamma \Rightarrow ?A, \Delta}$$

$$(\mathrm{L?})\ \frac{!\Gamma, A \Rightarrow ?\Delta}{!\Gamma, ?A \Rightarrow ?\Delta} \quad (\mathrm{R!})\ \frac{!\Gamma \Rightarrow A, ?\Delta}{!\Gamma \Rightarrow !A, ?\Delta} \quad (\mathrm{R?})\ \frac{\Gamma \Rightarrow A, \Delta}{\Gamma \Rightarrow ?A, \Delta} \quad (\mathrm{L!})\ \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, !A \Rightarrow \Delta}$$